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Underlying Probability Distributions of Planck's Radiation Law

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The derivation of Planck's radiation law can be considered as a transformation of a thermodynamic relation for black-body radiation into a fundamental relation in which the error law is the negative binomial distribution. In both limiting frequency ranges it transforms into Poisson distributions; in the Wien limit, it is the distribution of the number of photons, whose most probable value is given by Boltzmann's expression, while in the Rayleigh-Jeans limit, it is the distribution of the number of Planck oscillators. In the general case, they are Bernoullian random variables. In the Rayleigh-Jeans limit, the probability of determining the number of oscillators in a given frequency interval for a fixed value of the energy can be inverted to determining the probability of the energy for a fixed number of oscillators. The probability density is that of the canonical ensemble.

1. THERMODYNAMICS OF BLACK-BODY RADIATION

It is always intriguing to wonder if a great scientific discovery could have been overlooked had the discoverer used all the knowledge that was available at the time. Alternatively, knowing too much can prevent the step that is necessary to break with the past. A case in point is Planck's (1900) discovery of the formula of black-body radiation. When he wrote down his formula, he was not at all concerned about the fact that it contradicted the law of equipartition in all but the long-wavelength limit. He probably did not know the law existed (Klein, 1977). Surely, Lord Rayleigh (1905) was well aware of the law of equipartition, but could not pass judgement on Planck's formula because he could not follow its derivation. It was Einstein (1905), several years afterward, who fully appreciated the undermining of the laws of classical physics that went into Planck's derivation of the radiation formula.

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At the time of Planck's derivation, several facts about black-body radiation were known. Boltzmann had devised a proof of Stefan's law,

$$\bar{U} = bVT^4 \tag{1}$$

based on a Carnot cycle in which radiation played the role of the working substance. Expression (1) relates the (average) energy $\bar{U}$ to the fourth power of the absolute temperature T, where V is the volume and $cb/4$ is the Stefan-Boltzmann constant, with c the velocity of light in vacuum.

It was also known that radiation, like aerial vibrations, exerts a pressure equal to *one-third* the energy density

$$p = \bar{u}/3 \tag{2}$$

Boltzmann had assumed (2) by reasoning that the pressure in any particular direction should only be one-third of the energy density because space is isotropic. Introducing these two equations of state into the Euler relation,

$$s = \frac{1}{T}\bar{u} + \frac{p}{T} \tag{3}$$

leads at once to the fundamental relation

$$s = \tfrac{4}{3}b^{1/4}\bar{u}^{3/4} \tag{4}$$

for the entropy density.

Although (4) is thermodynamically admissible, since the entropy is extensive and a concave function of the energy and volume, it is not the most "fundamental" relation there is for the entropy. In order for the relation to be the most fundamental one, the entropy must be expressed as a logarithm, like that of an ideal gas

$$s = \bar{n}\log(\bar{u}^a\bar{n}^b) \tag{5}$$

where $\bar{n}$ is the (average) particle density.

Considering s as a function of $\bar{u}$ only, the law of error leading to the average energy density as the most probable value of the energy measured is

$$f(u) = \exp\{s(u) - s(\bar{u}) - s'(\bar{u})(u - \bar{u})\} \tag{6}$$

where the prime means differentiation. The Gaussian error law (6) can be written in the canonical form

$$f(u \mid \beta) = e^{-\beta u}\Omega(u)/\mathscr{Z}(\beta) \tag{7}$$

which shows that it belongs to the exponential family where

$$s'(\bar{u}) = \beta \tag{8}$$

is the inverse temperature in energy units where Boltzmann's constant is unity.

For continuous values of u, (7) is the probability density of the canonical ensemble. Moreover, it has precisely the form of Neyman's factorization theorem for the existence of a sufficient statistic, viz., the sample mean energy is a sufficient statistic for estimating the inverse temperature (Mandelbrot, 1962). The norming factor $\mathscr{Z}(\beta)$ is known as the partition function. The prior density, or structure function $\Omega(u)$ completely determines the mechanical structure of the isolated system before it is converted into a closed system by sampling the energy. At most, $\Omega(u)$ increases only as a finite, fixed power of u (Blanc-Lapierre and Tortat, 1956), which requires a fundamental relation of the form (5). This is not fulfilled by the fundamental relation of black-body radiation (4), which should have immediately been interpreted as implying the existence of another, more fundamental relation of the form (5).

2. STATISTICS OF BLACK-BODY RADIATION

Planck's derivation of his radiation formula can be shown to be an attempt to convert the fundamental relation (4) into a logarithmic form so that it would correspond to a law of error for an extensive variable. Planck had two other pieces of information: The first was Wien's displacement law

$$\bar{u}_\nu(\beta) = \nu^3 g(\beta\nu) \tag{9}$$

where $\bar{u}_\nu \, d\nu$ is the energy density of radiation lying in the frequency interval from ν to $\nu + d\nu$. Here $g(\beta\nu)$ is a function of the single variable $\beta\nu$, so that the spectral distribution (9) is determined for all temperatures once it is known for a single temperature. This function must be such that when (9) is integrated over all frequencies, it reproduces the Stefan–Boltzmann law (1). The second piece of information was the Wien formula

$$\bar{u}_\nu^{(\mathrm{Wien})}(\beta) = \alpha\nu^3 \exp(-\beta\gamma\nu) \tag{10}$$

for the spectral distribution, where α and γ are constants.

Assuming that the Euler relation (3) is valid for each mode of the electromagnetic field (Lavenda and Dunning-Davies, 1990*b*), we have

$$s(\bar{u}_\nu) = \beta(\bar{u}_\nu + p_\nu) \tag{11}$$

for each mode ν. The pressure exerted by a mode p_ν can be determined by solving Wien's formula (10) for β

$$s'^{\mathrm{Wien}}(\bar{u}_\nu) = \beta = -\frac{1}{\gamma\nu} \log\left(\frac{\bar{u}_\nu}{\alpha\nu^3}\right) \tag{12}$$

and integrating it to obtain

$$s^{\text{Wien}}(\bar{u}_\nu) = -\frac{\bar{u}_\nu}{\gamma\nu}\left[\log\left(\frac{\bar{u}_\nu}{\alpha\nu^3}\right) - 1\right] \tag{13}$$

From (12) it follows that Wien's law only holds for weak intensities, $\bar{u}_\nu < \alpha\nu^3$. Comparing (11) and (13), we obtain

$$p_\nu^{\text{Wien}} = \frac{\bar{u}_\nu}{\beta\gamma\nu} \tag{14}$$

which, when integrated over all frequencies, gives the radiation pressure (2). The Wien pressure per mode as a function of frequency is shown in Figure 1.

Expression (13) is a fundamental relation and should correspond to a law of error identifying the average value of the energy as the most probable value of the energy that is measured. On substituting the entropy (13) into the error law (6), we obtain the expressions

$$\Omega(u) = \left(\frac{u}{\alpha\nu^3}\right)^{-u/\gamma\nu} e^{u/\gamma\nu} \tag{15}$$

and

$$\log \mathscr{Z}(\beta) = \frac{\alpha\nu^2}{\gamma} e^{-\beta\gamma\nu} \tag{16}$$

for the structure function and logarithm of the partition function, respectively. Hence it appears that we are no closer to a fundamental relation for black-body radiation than our original relation, (4).

The structure function (15) looks like Stirling's approximation for a factorial if only u could somehow be treated as a discrete variable. The logarithm of the partition function (16) would then be some average number density. In fact, by defining

$$\bar{n}_\nu \doteq \bar{u}_\nu/\gamma\nu, \qquad n \doteq u/\gamma\nu, \qquad \bar{m}_\nu \doteq \alpha\nu^2/\gamma \tag{17}$$

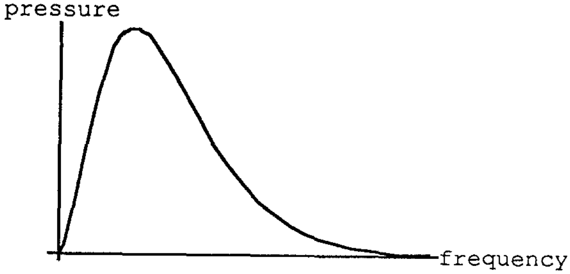

Fig. 1. The Wien pressure as a function of the frequency.

we can write the entropy density per mode (13) as

$$s^{\text{Wien}}(\bar{n}_\nu) = \log\left(\frac{\bar{m}_\nu^{\bar{n}_\nu}}{\bar{n}_\nu!}\right) \tag{18}$$

provided n is sufficiently large so that Stirling's formula can be applied. Now, the law of error is (Lavenda, 1988)

$$f(n) = \frac{\bar{n}_\nu^n}{n!}\exp(-\bar{n}_\nu) \tag{19}$$

which is precisely the Poisson distribution. The revolutionary step is contained in (17)—the assumption of discrete "particles"—and its justification resides in the derivation of Poisson distribution as the law of error identifying the average number of "particles" as the most probable value of the quantity measured.

Further support would have been forthcoming had (17) been introduced into Wien's formula (10) to obtain

$$\bar{n}_\nu = \bar{m}_\nu\, e^{-\beta\varepsilon_\nu} \tag{20}$$

where $\varepsilon_\nu = \gamma\nu$. Expression (20) is the most probable "distribution" of Boltzmann statistics, where $\bar{n}_\nu$ is the most probable, or equivalently, average number of particles of the group ν that is found in $\bar{m}_\nu$ cells. Equipartition of energy surely cannot be applied locally to each mode. Rather, the theorem of equipartition of energy applies to the average energy density

$$\bar{u} = 3\bar{n}/\beta \tag{21}$$

for a system with $6\bar{n}$ degrees of freedom per unit volume.

In addition, the first relation in (17) could have been introduced in the Wien expression for the pressure per mode (14), which would have resulted in

$$p_\nu = \bar{n}_\nu/\beta$$

which is the equation of state of an ideal gas. Integrating over all frequencies gives $\beta p = \bar{n}$, which in view of (21) gives the radiation pressure (2).

Hence, had Planck attempted to place Wien's formula on firm theoretical ground, he would have certainly been led to the discreteness hypothesis contained in (17) and found that its justification was simpler than the path he was to follow, since it involved Boltzmann statistics. Planck, however, was not aware of the limiting nature of the Poisson distribution and it was not apparent that (20) is only valid under the condition that $\bar{n}_\nu \ll \bar{m}_\nu$. This was to come out of the experimental investigations of Rubens and Kurlbaum, who showed that Wien's formula (10) broke-down in the far-infrared region, where $\bar{u}_\nu \propto T$, in accordance with the law of equipartition of energy. It is rather ironic that quantum theory was discovered only after "classical" deviations from the quantum regime had been observed in the far infrared.

The short-wavelength regime of black-body radiation points to the discreteness of "quanta" of energy,

$$\bar{u}_\nu = \bar{n}_\nu \varepsilon_\nu \tag{22}$$

they obey "classical" or Boltzmann statistics. This was emphasized in Einstein's (1905) paper on light quanta. It is precisely in the long-wavelength regime, where the law of equipartition of energy is obeyed, that the statistics is nonclassical. It is well known that Planck worked from the expression for $-(\partial^2 s/\partial \bar{u}_\nu^2)^{-1}$ in which he added a quadratic term in the energy, and integrating once and using the second law, he obtained his celebrated formula. In order to place his formula on firm theoretical ground, he had to take recourse to Boltzmann's probabilistic formulation of the entropy. We will proceed in a way which utilizes actual probability distributions rather than binomial coefficients.

Boltzmann statistics, like the Poisson distribution, is a limiting form that is valid in a well-defined limit. That limit was unknown to Planck at the time and it took another quarter of a century before it became clear what was happening in the passage to that limit. Historically, the Poisson distribution was known as "Poisson's limit law" to which the binomial distribution converges in the limit as the number of Bernoulli trials increases without limit while the probability of success tends to zero in such a way that their product is of moderate magnitude. This, together with the particle nature that emerges in the Wien limit, could have led Planck to explore the possibility of obtaining the actual probability distribution. However, Boltzmann's theory relating the entropy to what Planck later called the "thermodynamic probability," in order to distinguish it from a real probability distribution, does not deal with probability distributions, so Planck, following Boltzmann, could not have imagined the connection.

Classically, the probability that a "particle" is present is given by the Boltzmann factor $q = e^{-\beta\varepsilon_\nu}$. The probability of there not being one is simply $p = (1 - e^{-\beta\varepsilon_\nu})$. The probability that there are n particles is given by the geometric distribution (Planck, 1932)

$$f(n) = e^{-(n-1)\beta\varepsilon_\nu}(1 - e^{-\beta\varepsilon_\nu}) \tag{23}$$

which is valid for any mode in the frequency range between ν and $\nu + d\nu$. Now suppose we have $\bar{m}_\nu$ oscillators lying in this frequency interval. Since each of these oscillators is independent (Planck's assumption of "natural radiation"), the moment generating function will be $G^{\bar{m}_\nu}(z)$, where

$$G(z) = \frac{pz}{1 - qz}$$

is the moment generating function of the geometric distribution (23). On the strength of the binomial series expansion,

$$(1-qz)^{-\bar{m}_\nu} = \sum_{n=0}^{\infty} \binom{\bar{m}_\nu + n - 1}{n} (qz)^n$$

we have

$$\begin{aligned} G^{\bar{m}_\nu}(z) &= \sum_{n=0}^{\infty} \binom{\bar{m}_\nu + n - 1}{n} p^{\bar{m}_\nu} q^n z^{\bar{m}_\nu + n} \\ &= \sum_{k=\bar{m}_\nu}^{\infty} \binom{k-1}{\bar{m}_\nu - 1} p^{\bar{m}_\nu} q^{k-\bar{m}_\nu} z^k \end{aligned} \tag{24}$$

Therefore, the probability of finding n *indistinguishable* particles among the $\bar{m}_\nu$ oscillators with empty oscillators being admissible is given by the negative binomial distribution (Lavenda and Figueiredo, 1989)

$$f(n) = \binom{\bar{m}_\nu + n - 1}{n} e^{-n\beta\varepsilon_\nu}(1 - e^{-\beta\varepsilon_\nu})^{\bar{m}_\nu} \tag{25}$$

If we impose the constraint that particle number can never be inferior to the number of oscillators, the negative binomial distribution (25) is replaced by the Pascal distribution,

$$f(n) = \binom{n-1}{\bar{m}_\nu - 1} e^{-(n-\bar{m}_\nu)\beta\varepsilon_\nu}(1 - e^{-\beta\varepsilon_\nu})^{\bar{m}_\nu}, \qquad n \geq \bar{m}_\nu \tag{26}$$

as shown by formula (24). The radiation laws that are derived from (25) and (26) differ by an integral zero-point energy term (Lavenda and Figueiredo, 1989) and this excludes the high-frequency end of the spectrum in (26).

Casting the negative binomial distribution (25) as the Gaussian error law,

$$f(n) = \exp\{s(n) - s(\bar{n}_\nu) - s'(\bar{n}_\nu)(n - \bar{n}_\nu)\}$$

gives the expression

$$s^{\text{Planck}}(\bar{u}_\nu) = \beta\bar{u}_\nu - \bar{m}_\nu \log(1 - e^{-\beta\varepsilon_\nu}) \tag{27}$$

for the entropy density per mode. Comparing (27) with the Euler relation (11) determines the pressure

$$p_\nu^{\text{Planck}} = -\frac{\bar{m}_\nu}{\beta} \log(1 - e^{-\beta\varepsilon_\nu}) \tag{28}$$

which goes over into the Wien expression (14) for $\beta\varepsilon_\nu \gg 1$. The pressure as a function of frequency is shown in Figure 2.

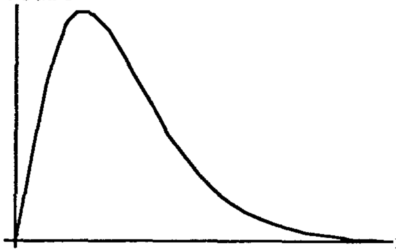

Fig. 2. The Planck pressure as a function of the frequency.

Since $-p_\nu$ is equal to the Helmholtz free energy density per mode, differentiating $-\beta p_\nu$ with respect to β yields

$$\bar{u}_\nu^{\text{Planck}} = \frac{\bar{m}_\nu \varepsilon_\nu}{e^{\beta\varepsilon_\nu} - 1} \tag{29}$$

Then, consulting the first definition in (17) leads to

$$\bar{n}_\nu^{\text{Planck}} = \frac{\bar{m}_\nu}{e^{\beta\varepsilon_\nu} - 1} \tag{30}$$

which is Planck's "distribution." In the limit $\beta\varepsilon_\nu \gg 1$, it transforms into the Boltzmann distribution (20).

Solving (29) for β and either integrating the second law (8) or substituting it directly into (27) gives

$$s^{\text{Planck}}(\bar{u}_\nu) = \frac{\bar{u}_\nu}{\varepsilon_\nu} \log\left(\frac{\bar{u}_\nu + \bar{m}_\nu \varepsilon_\nu}{\bar{u}_\nu}\right) + \bar{m}_\nu \log\left(\frac{\bar{u}_\nu + \bar{m}_\nu \varepsilon_\nu}{\bar{m}_\nu \varepsilon_\nu}\right) \tag{31}$$

which, according to (17), can be expressed in terms the average particle density as

$$s^{\text{Planck}}(\bar{n}_\nu) = \bar{n}_\nu \log\left(\frac{\bar{m}_\nu + \bar{n}_\nu}{\bar{n}_\nu}\right) + \bar{m}_\nu \log\left(\frac{\bar{m}_\nu + \bar{n}_\nu}{\bar{m}_\nu}\right) \tag{32}$$

Planck, having his expression for the spectral distribution in hand, went in the opposite direction: He solved (29) for β and integrated to get (31). Then, in order to use Boltzmann's principle, he had to count the number of complexions. This forced Planck to "treat [$\bar{u}_\nu$] not as a continuous, infinitely divisible quantity, but rather as composed of an integral number of equal finite parts" (Planck, 1900).

It is rather interesting to note that Planck identified the entropy with the logarithm of

$$\binom{\bar{m}_\nu + n - 1}{n}$$

where n is a random variable. In order to compare it with his expression (31), he had to replace n by $\bar{n}_\nu$. This has the effect of maximizing the stochastic entropy density $s(n)$ to give the thermodynamic entropy density $s(\bar{n}_\nu)$. It is precisely the fact that the average value is also the most probable value that is what is behind the success of Planck's analysis.

3. PARTICLES VERSUS OSCILLATORS

Consider a sequence of balls separated by bars (Ehrenfest and Kamerlingh Onnes, 1914). Two consecutive bars indicate a cell. If there are n balls and m cells, there will be $m+1$ bars, but since the first and last bar must always be fixed, only $m-1$ are movable. These bars together with the n balls can appear in any order, so that the number of distinguishable distributions equals the number of ways of selecting n places out of $m+n-1$, namely the binomial coefficient in (25). Denote by $\eta = m+n$ the total number of balls and cells. For a fixed number of cells, η will vary because n does. But now suppose that we fix η and allow m to vary; in other words, the numbers of cells and balls may vary, but in such a way that their sum remains constant.

First, suppose that $m \gg n$. Eliminating m in favor of η in (25) gives

$$f(n) = \binom{\eta}{n} e^{-n\beta\varepsilon_\nu}(1-e^{-\beta\varepsilon_\nu})^{\eta-n} \tag{33}$$

since $\eta \gg 1$. Expression (33) is the binomial distribution, which, in the limit as $e^{-\beta\varepsilon_\nu} \to 0$ and $\eta \to \infty$ such that their product

$$\bar{n}_\nu = \eta\, e^{-\beta\varepsilon_\nu} \tag{34}$$

is a constant of moderate size, transforms into the Poisson distribution (19). This is the Wien limit, where (34) is essentially expression (20).

Second, consider the opposite limit, $n \gg m$. Eliminating n may in favor of η in (25) gives

$$f(m) = \binom{\eta}{m} e^{-(\eta-m)\beta\varepsilon_\nu}(1-e^{-\beta\varepsilon_\nu})^{m} \tag{35}$$

which is again a binomial distribution. In the limit as $(1-e^{-\beta\varepsilon_\nu}) \approx \beta\varepsilon_\nu \to 0$ and $\eta \to \infty$ such that

$$\beta\eta\varepsilon_\nu = \bar{m}_\nu \tag{36}$$

is a constant of moderate magnitude, the binomial distribution (35) transforms into the Poisson distribution

$$f(m) = \frac{\bar{m}_\nu^m}{m!}\, e^{-\bar{m}_\nu} \tag{37}$$

whose entropy density is

$$s^{\mathrm{RJ}}(\bar{m}_\nu)=\log\left(\frac{\bar{n}_\nu^{\bar{m}_\nu}}{\bar{m}_\nu!}\right) \tag{38}$$

Expression (36) is essentially the law of equipartition of energy and (38) has been derived from the distribution (35) in the limit $\beta\varepsilon_\nu \ll 1$.

The physical picture which emerges is that of an entity which can be in one of two forms: either a particle or an oscillator. These are Bernoullian random variables which are governed by the binomial distributions (33) and (35), respectively. In both the Wien and Rayleigh-Jeans limits, one of the two forms predominates. In the former (latter) limit, the appearance of a particle (oscillator) is a rare event which is governed by the "law of small numbers" or the Poisson distribution (19) [(37)]. The granularity in the number of oscillators appears in the Rayleigh-Jeans limit in exactly the same way that the granularity in the number of photons occurs in the Wien limit.

We may use an argument (Lavenda and Dunning-Davies, 1991), similar to that used by Einstein (1905) in support of the particle nature of light in the Wien limit, to show that the same holds true for the oscillators in the Rayleigh-Jeans limit. We want to determine the probability that in a small volume V of a much larger volume V_0 in which there are N_0 gas particles which are uniformily distributed, there will be exactly N particles. This is given by the binomial distribution

$$f(N)=\binom{N_0}{N}\left(\frac{V}{V_0}\right)^N\left(1-\frac{V}{V_0}\right)^{N_0-N} \tag{39}$$

In the limit as $N_0\to\infty$ and $V_0\to\infty$ such that their ratio

$$N_0/V_0=\bar{N}/V \tag{40}$$

is finite, where $\bar{N}$ is the average number of particles in the volume V, the binomial distribution (39) transforms into the Poisson distribution

$$f(N)=\frac{\bar{N}^N}{N!}e^{-\bar{N}}$$

whose entropy is (Lavenda and Dunning-Davies, 1991)

$$S(\bar{N})=-\bar{N}\left[\log\left(\frac{\bar{N}}{N_0}\right)-1\right] \tag{41}$$

With the aid of the homogeneity condition (40), the entropy (41) can be written as

$$S(\bar{N})=-\bar{N}\left[\log\left(\frac{V}{V_0}\right)-1\right] \tag{42}$$

In the Wien limit, the entropy (42) must be the same as V_0 times the entropy density (18); this requires

$$\bar{N}/N_0 = V/V_0 = \bar{n}_\nu/\bar{m}_\nu \tag{43}$$

and

$$\bar{N} = \bar{n}_\nu V_0 \tag{44}$$

The average number of particles in the subvolume is thus identified with the average number of photons present in the cavity. Introducing (44) into (43) gives $N_0 = \bar{m}_\nu V_0$, which associates the total number of particles with the number of modes per unit frequency interval.

Alternatively, in the Rayleigh-Jeans limit, (42) must coincide with V_0 times the entropy density (38); this demands

$$\bar{N}/N_0 = V/V_0 = \bar{m}_\nu/\bar{n}_\nu \tag{45}$$

and

$$\bar{N} = \bar{m}_\nu V_0 \tag{46}$$

The average number of particles in the subvolume is now to be identified with the average number of oscillators per unit frequency interval in the cavity. Eliminating $\bar{N}$ between (45) and (46) leads to $N_0 = \bar{n}_\nu V_0$, which associates the total number of particles with the number of photons per unit frequency interval in the cavity. It is quite evident from (44) and (46) that the roles of the photons and oscillators have been interchanged.

The binomial distribution (35) furnishes two expressions for the entropy: the statistical entropy density, or the maximum of the logarithm of the binomial coefficient,

$$s(\bar{m}_\nu) = -\eta \log\left(\frac{\eta - \bar{m}_\nu}{\eta}\right) - \bar{m}_\nu \log\left(\frac{\eta - \bar{m}_\nu}{\bar{m}_\nu}\right)$$

and the thermodynamic entropy,

$$s(\bar{m}_\nu) = \beta u - \bar{m}_\nu \log(e^{\beta\varepsilon_\nu} - 1) \tag{47}$$

where the total energy, $\eta\varepsilon_\nu = u$, is constant. Equating their derivatives with respect to $\bar{m}_\nu$ gives the average number of oscillators in the frequency interval $d\nu$ as

$$\bar{m}_\nu \, d\nu = \eta(1 - e^{-\beta\varepsilon_\nu}) \, d\nu \tag{48}$$

which is equivalent to 30. Comparing (47) to the Euler relation (11) gives

$$p_\nu = -\frac{\bar{m}_\nu}{\beta} \log(e^{\beta\varepsilon_\nu} - 1) \tag{49}$$

for the pressure, which is shown in Figure 3.

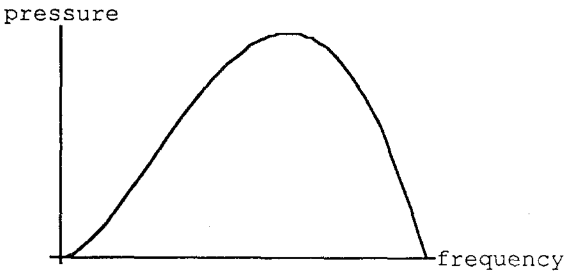

Fig. 3. The low-frequency pressure as a function of frequency.

The area under the curve is one-third the total energy density. The pressure curve shown in Figure 3 stands on par with the pressure distribution shown in Figure 1. The underlying probability distributions are both discrete and describe, respectively, the distribution of the number of oscillators and particles in the low- and high-frequency ranges of the spectrum. The energy density per mode is an unbounded function of the frequency, while the pressure per mode passes through a maximum and vanishes at

$$\nu_{\mathrm{th}} = \frac{1}{\beta_\gamma} \log 2 \tag{50}$$

which is the threshold frequency that separates the wave from the particle nature of light (Lavenda and Figueiredo, 1989). Numerically, $\nu = 0.14\nu_{\max}$, where $\nu_{\max}$ is the frequency at which $\bar{u}_\nu$ is maximum. For a temperature of 600 K, ν_{th} lies on the border between the far and near infrared. This threshold frequency is related to the maximum uncertainty of finding a photon in a given mode (Brillouin, 1962),

$$e^{-\beta\gamma\nu_{\mathrm{th}}} = 1 - e^{-\beta\gamma\nu_{\mathrm{th}}} = \tfrac{1}{2}$$

Integrating the pressure (49) over frequencies up to ν_{th} gives the radiation pressure (2).

4. THE CLASSICAL LIMIT

In the low-frequency limit, $\beta\varepsilon_\nu \ll 1$, the exponential in (28) may be expanded to first order; using the law of equipartition of energy

$$\beta\bar{u}_\nu = \bar{m}_\nu \tag{51}$$

leads to

$$p_\nu^{\mathrm{RJ}} = -\frac{\bar{m}_\nu}{\beta} \log\left(\frac{\bar{m}_\nu \varepsilon_\nu}{\bar{u}_\nu}\right) \tag{52}$$

The Rayleigh–Jeans limit is the high-intensity limit $\bar{u}_\nu > \alpha\nu^3$ which is required in order that the pressure per mode (52) be positive. Introducing both equations of state (52) and (51) into the Euler relation (11) gives the fundamental relation

$$s^{\mathrm{RJ}}(\bar{u}_\nu) = \bar{m}_\nu \left[\log\left(\frac{\bar{u}_\nu}{\alpha\nu^3}\right) + 1 \right]$$
$$= \log\left[\frac{(\bar{u}_\nu/\varepsilon_\nu)^{\bar{m}_\nu}}{\bar{m}_\nu!}\right] \tag{53}$$

provided $\bar{m}_\nu$ is large enough to permit the use of Stirling's approximation. Expression (53) relates the entropy to the logarithm of the phase volume occupied by the system. With the aid of (22), it can be written as (38). A comparison with the Wien entropy (18) shows that $\bar{m}_\nu$ and $\bar{n}_\nu$ have traded roles. There is no "classical" analog to such an expression; nevertheless, it is a direct consequence of the "classical" limit where equipartition of energy (51) applies. By contrast with the Wien limit, "particle" discreteness is replaced by "mode" discreteness. Moreover, Planck's (1899) calculation of the number of electromagnetic modes per unit volume having frequencies in the interval between ν and $\nu + d\nu$, $\bar{m}_\nu\, d\nu = (8\pi\nu^2/c^3)\, d\nu$ should be interpreted as the *average* number of oscillators in this interval.

Introducing (53) into the error law (6) leads to

$$f(u) = \frac{(\beta u)^m}{m!} e^{-\beta u} \tag{54}$$

But this cannot be the law of error for the energy, because it is not normalized. Rather, it is the Poisson distribution (37) where the scale parameter β is the expected number of oscillators in a given frequency interval per unit energy. The existence of another probability distribution is implied by the fact that β is *not* the scale parameter for the distribution (54) (Lavenda and Dunning-Davies, 1990*a*). In Lavenda and Dunning-Davies (1990*a*) it was assumed that the number of degrees of freedom was a random number whose distribution is (54). Here, we have derived it from the error law (6) in the long-wavelength or high-intensity limit of black-body radiation.

The extension-in-phase, or phase space volume, is specified by the energy u (Gibbs, 1902). If the degrees of freedom are "well mixed," their distribution, for a fixed value of the energy u, is given by the Poisson distribution (54) in the long-wavelength limit of black-body radiation. Suppose we want to determine the phase volume that contains exactly $2\bar{m}_\nu$ degrees of freedom. This can be done by varying the energy u until the phase volume contains exactly this number of degrees of freedom. The energy $\tilde{u}$ of the phase volume that contains $2\bar{m}_\nu$ degrees of freedom is the required random variable. The probability that the phase volume will contain less than $2\bar{m}_\nu$ degrees of freedom is

$$\Pr(\tilde{u} > u) = \sum_{j=0}^{\bar{m}_\nu - 1} \frac{(\beta u)^j}{j!} e^{-\beta u} = \int_u^\infty \frac{(\beta u')^{\bar{m}_\nu - 1}}{(\bar{m}_\nu - 1)!} \beta e^{-\beta u'} du'$$

where the second equality is established by $\bar{m}_\nu - 1$ integration by parts. Since the probability $\Pr(\tilde{u} \leq u) = 1 - \Pr(\tilde{u} > u)$ is the cumulative distribution function $F(u \mid \beta)$ of the random variable $\tilde{u}$, the probability density is

$$f(u \mid \beta) = \frac{\partial F}{\partial u} = \frac{(\beta u)^{\bar{m}_\nu - 1}}{\Gamma(\bar{m}_\nu)} \beta e^{-\beta u} \tag{55}$$

where Γ is the gamma function. Thus, the random variable $\tilde{u}$ has a probability density function given by (55), which is a chi-square distribution with $2\bar{m}_\nu$ degrees of freedom. This is precisely Gibbs' density of the canonical ensemble (7).

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